Combinatorial Manifolds: Bonnet-Myer's Theorem (RTG)

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Our Goal: Combinatorial Bonnet Meyer's Theorem

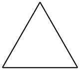
THM: A combinatorial 3-manifold with edges of degree at most five has edge diameter at most five.

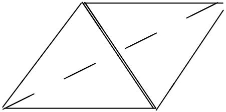
Simplex

A **simplex** is a generalization of a triangle into different dimensions:

0-simplex: •

1-simplex: ——

2-simplex: 

3-simplex: 

Simplicial Complex

A **face** of an n-simplex is an $(n-1)$-simplex.

A **simplicial complex** is a set of simplicies such that:

- Faces of simplices in the complex are also in the complex.
- Intersection of any two simplices is a face of both of them.

Star and Link of a Simplex

The **star** of a simplex is the set of other simplicies with faces in the simplex.

The **closure of the star** of a simplex is the smallest simplicial complex containing all the simplicies of the star.

The **link** of a simplex is the closure of its star minus its star.

Positively Curved Combinatorial Manifold

A **(Boundaryless) Combinatorial** n**-manifold**, M^n, is a simplicial complex in which the link of each k-simplex is an $(n-k-1)-$sphere.

M^n is **positively curved** if each $(n-2)$-simplex has degree at most

$$\epsilon(n) = \begin{cases} 5, & n = 2, 3 \\ 4, & n \geq 4 \end{cases}$$

Curvature and degree bounds

Curvature is the measure of the flatness of a space.

Positive curvature requires the total angle around each $(n-2)$-simplex σ be less than 2π.

Example: $n = 3$
Total angle around each edge(1-simplex) σ is
$\deg(\sigma)\cos^{-1}(\frac{1}{3}) \approx \deg(\sigma)1.23 < 2\pi \approx 6.28$.
Therefore, $\deg(\sigma) < 5.10$.

Hops

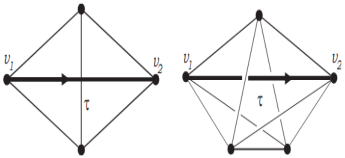


Figure 1: Two and three dimensional hops

$$H_n = \sqrt{2 + \frac{2}{n}}$$

Jumps

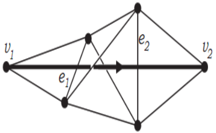


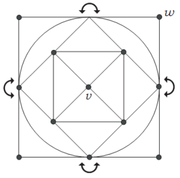
Figure 2: A jump

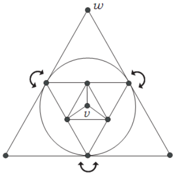
$$J = \sqrt{3} + \frac{1}{2}\sqrt{2}$$

Combinatorial Bonnet Meyer's Theorem

THM: A combinatorial 3-manifold with edges of degree at most five has edge diameter at most five.

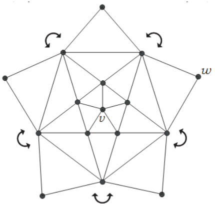
Positively Curved Combinatorial 2-manifolds





Octahedron

Tetrahedron



Icosahedron

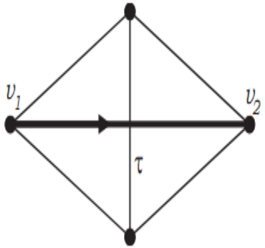
Distance between Vertices in M^2

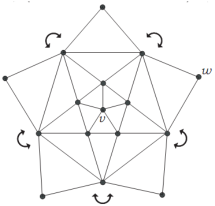
$d(v, w)$ = minimum distance between vertices v, w.

$d_1(v, w)$ = minimum edge distance between vertices v, w.

$d(v, w) \in \{0, 1, H_2, 1 + H_2\}$ for any vertices $v, w \in M^2$.

$d(v, w) = 1 + H_2$ if and only if $d_1(v, w) = 3$.

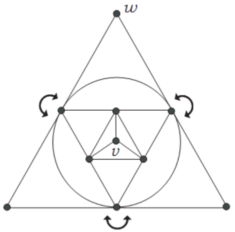




Another Useful Result in M^2

For a fixed vertex v in M^2, we have $d_1(v, w) = 3$ for at most one vertex w.

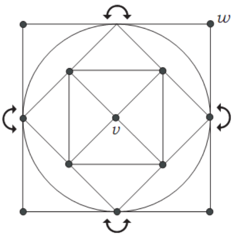
Example: Look at the tetrahedron model.



Minimal Paths and a 2D result

A **Minimal Path** is a path with length equal to that of the distance between the endpoints of the path.

Suppose $d_1(v, w) = 3$ for vertices v, w in a positively curved surface M^2. Then any minimal hop beginning on v ends on a vertex in $Lk(w)$.



3-Dimensional Case

Using the links of vertices in M^3, we reduce to an M^2 case and find results such as:

$d(v, w) \in \{0, 1, H_3, 2, J, 3, 2H_3, 4, 2J\}$ for any vertices $v, w \in M^3$
$d(v, w) = 2J \Rightarrow d_1(v, w) = 5$

These give us the result of the theorem we desired:

THM: A combinatorial 3-manifold with edges of degree at most five has edge diameter at most five.

Further Thoughts:

Next Step: Generalize results for edge lengths of a general metric.

Definite Changed:

- Measurements of hops and jumps.
- Minimal Path lengths.

Possibly Unchanged:

- Path types: using edges, hops, and jumps.
- Assumptions about categorizing 2-dimensional manifolds.

Acknowledgements

Andrea Young for advising me.

The paper we were investigating: Positively Curved Combinatorial 3-Manifolds by Aaron Trout.

Wikipedia for some definitions.